On Sarnak's saturation problem

SL Rydin Myerson
tinyurl.com/MPIMlunchseminar

MPIM Bonn
2026/07/22

On Sarnak's saturation problem

A set of rational points has finite saturation number, in the sense of Sarnak, if it has a Zariski dense subset consisting of points whose co-ordinates have a bounded number of terms in their prime factorisations.

On Sarnak's saturation problem

A set of rational points has finite saturation number, in the sense of Sarnak, if it has a Zariski dense subset consisting of points whose co-ordinates have a bounded number of terms in their prime factorisations.

Do the rational points of an irreducible variety always have finite saturation number? Or is there an irreducible variety whose rational points do *not* have finite saturation number?

On Sarnak's saturation problem

A set of rational points has finite saturation number, in the sense of Sarnak, if it has a Zariski dense subset consisting of points whose co-ordinates have a bounded number of terms in their prime factorisations.

Do the rational points of an irreducible variety always have finite saturation number? Or is there an irreducible variety whose rational points do *not* have finite saturation number?

We will see that, conjecturally, there is. We prove several weaker versions of this claim, and suggest some ideas for future work.

Rational points

Write points $\mathbf{X} \in \mathbb{Q}^s$ with rational co-ordinates in s-dimensional space as

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \ldots, x_s/x_{s+1}),$$

where $\gcd(x_1, \ldots, x_{s+1}) = 1$ and $x_{s+1} > 0$; we call $\mathbf{x} = (x_1, \ldots, x_{s+1})$ the vector of *homogeneous co-ordinates*.

Rational points

Write points $\mathbf{X} \in \mathbb{Q}^s$ with rational co-ordinates in s-dimensional space as

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \ldots, x_s/x_{s+1}),$$

where $\gcd(x_1, \ldots, x_{s+1}) = 1$ and $x_{s+1} > 0$; we call $\mathbf{x} = (x_1, \ldots, x_{s+1})$ the vector of *homogeneous co-ordinates*.

The *integer points* $\mathbf{X} \in \mathbb{Z}^s$ are the points with $x_{s+1} = 1$. If S is a set of primes, the S*-integer points* $\mathbf{X} \in \mathbb{Z}[S^{-1}]^s$ are the points for which x_{s+1} is divisible only by primes in S.

Rational points

Write points $\mathbf{X} \in \mathbb{Q}^s$ with rational co-ordinates in s-dimensional space as

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \dots, x_s/x_{s+1}),$$

where $\gcd(x_1, \dots, x_{s+1}) = 1$ and $x_{s+1} > 0$; we call $\mathbf{x} = (x_1, \dots, x_{s+1})$ the vector of *homogeneous co-ordinates*.

The *integer points* $\mathbf{X} \in \mathbb{Z}^s$ are the points with $x_{s+1} = 1$. If S is a set of primes, the S-*integer points* $\mathbf{X} \in \mathbb{Z}[S^{-1}]^s$ are the points for which x_{s+1} is divisible only by primes in S.

E.g.

$$\mathbb{Z}[\{2\}^{-1}]^2 = \{(a/2^n, b/2^n) : a, b, n \in \mathbb{Z}, \ n \geq 0\}$$

is the set of points with homogeneous co-ordinates $(a, b, 2^n)$.

Varieties

Write points $\mathbf{X} \in \mathbb{Q}^s$ in terms of *homogeneous co-ordinates* $\mathbf{x} = (x_1, \ldots, x_{s+1})$,

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \ldots, x_s/x_{s+1}),$$

where $\gcd(x_1, \ldots, x_{s+1}) = 1$ and $x_{s+1} > 0$.

Varieties

Write points $\mathbf{X} \in \mathbb{Q}^s$ in terms of *homogeneous co-ordinates* $\mathbf{x} = (x_1, \ldots, x_{s+1})$,

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \ldots, x_s/x_{s+1}),$$

where $\gcd(x_1, \ldots, x_{s+1}) = 1$ and $x_{s+1} > 0$.

Any system of polynomial equations with rational coefficients,

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0,$$

can be written in the form $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$ where the f_i are *homogeneous* with integer coefficients.

Varieties

Write points $\mathbf{X} \in \mathbb{Q}^s$ in terms of *homogeneous co-ordinates* $\mathbf{x} = (x_1, \ldots, x_{s+1})$,

$$\mathbf{X} = \mathbf{x}'/x_{s+1} = (x_1/x_{s+1}, \ldots, x_s/x_{s+1}),$$

where $\gcd(x_1, \ldots, x_{s+1}) = 1$ and $x_{s+1} > 0$.

Any system of polynomial equations with rational coefficients,

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0,$$

can be written in the form $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$ where the f_i are *homogeneous* with integer coefficients.

We say that the f_i (or F_i) define a *variety* V, and the rational solutions $\mathbf{X}$ to

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

are the *rational points* of V (with homogeneous co-ordinates $\mathbf{x}$ where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$).

Varieties

Any system of polynomial equations with rational coefficients,

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0,$$

can be written in the form $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$ where the f_i are *homogeneous* with integer coefficients.

We say that the f_i (or F_i) define a *variety* V, and the rational solutions $\mathbf{X}$ to

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

are the *rational points* of V (with homogeneous co-ordinates $\mathbf{x}$ where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$).

Varieties

Any system of polynomial equations with rational coefficients,

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0,$$

can be written in the form $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$ where the f_i are *homogeneous* with integer coefficients.

We say that the f_i (or F_i) define a *variety* V, and the rational solutions $\mathbf{X}$ to

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

are the *rational points* of V (with homogeneous co-ordinates $\mathbf{x}$ where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) = 0$).

Similarly, $V(\mathbb{Z})$ is the set of integral solutions to

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0,$$

and $V(\mathbb{Z}[S^{-1}])$ is the set of S-integral solutions.

Zariski topology

The *Zariski topology* is defined by taking all sets of the form

$$F_1(\mathbf{X}), \dots, F_m(\mathbf{X}) = 0$$

as the closed sets.

Zariski topology

The *Zariski topology* is defined by taking all sets of the form

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

as the closed sets.

So, if A is a subset of $\mathcal{O}$, then to say "A is Zariski dense in $\mathcal{O}$" means that any polynomial with rational coefficients which vanishes on A, must also vanish on all of $\mathcal{O}$.

Zariski topology

The *Zariski topology* is defined by taking all sets of the form

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

as the closed sets.

So, if A is a subset of $\mathcal{O}$, then to say "A is Zariski dense in $\mathcal{O}$" means that any polynomial with rational coefficients which vanishes on A, must also vanish on all of $\mathcal{O}$.

A variety V is *irreducible* if it cannot be written as a union of closed sets none of which contain V. Every variety is a union of finitely many irreducible components.

Zariski topology

The *Zariski topology* is defined by taking all sets of the form

$$F_1(\mathbf{X}), \ldots, F_m(\mathbf{X}) = 0$$

as the closed sets.

So, if A is a subset of $\mathcal{O}$, then to say "A is Zariski dense in $\mathcal{O}$" means that any polynomial with rational coefficients which vanishes on A, must also vanish on all of $\mathcal{O}$.

A variety V is *irreducible* if it cannot be written as a union of closed sets none of which contain V. Every variety is a union of finitely many irreducible components.

E.g. $X(Y-1) = 0$ is reducible, being the union of $X = 0$ and $Y = 1$.

Prime factors

Let $\Omega(x_1 \cdots x_{s+1})$ be the total number of prime factors of the homogeneous co-ordinates, counted with multiplicity. Set $\Omega(0) = \infty$.

Prime factors

Let $\Omega(x_1 \cdots x_{s+1})$ be the total number of prime factors of the homogeneous co-ordinates, counted with multiplicity. Set $\Omega(0) = \infty$.

(For the number of prime factors without multiplicity write $\omega(x_1 \cdots x_{s+1})$.)

Prime factors

Let $\Omega(x_1 \cdots x_{s+1})$ be the total number of prime factors of the homogeneous co-ordinates, counted with multiplicity. Set $\Omega(0) = \infty$.

(For the number of prime factors without multiplicity write $\omega(x_1 \cdots x_{s+1})$.)

E.g. the points of $X(Y-1) = 0$ with $\Omega(XY) \leq r$ must all have $X \neq 0$ and hence $Y = 1$. This funny behaviour is why we restrict to irreducible varieties.

Saturation

Let $\mathcal{O}$ be a set of rational points in s-dimensional space. We say that $\mathcal{O}$ *saturates* or has *finite saturation* if, for large enough r, the points of $\mathcal{O}$ with

$$\Omega(x_1 \cdots x_{s+1}) \leq r$$

are Zariski dense in $\mathcal{O}$. The smallest such r is called the *saturation number*.

Saturation

Let $\mathcal{O}$ be a set of rational points in s-dimensional space. We say that $\mathcal{O}$ *saturates* or has *finite saturation* if, for large enough r, the points of $\mathcal{O}$ with

$$\Omega(x_1 \cdots x_{s+1}) \leq r$$

are Zariski dense in $\mathcal{O}$. The smallest such r is called the *saturation number*.

That is: $\mathcal{O}$ saturates unless, for every r, there is some polynomial which doesn't vanish on the whole of $\mathcal{O}$, but which vanishes on all the points of $\mathcal{O}$ with

$$\Omega(x_1 \cdots x_{s+1}) \leq r.$$

Saturation

Let $\mathcal{O}$ be a set of rational points in s-dimensional space. We say that $\mathcal{O}$ *saturates* or has *finite saturation* if, for large enough r, the points of $\mathcal{O}$ with

$$\Omega(x_1 \cdots x_{s+1}) \leq r$$

are Zariski dense in $\mathcal{O}$. The smallest such r is called the *saturation number*.

That is: $\mathcal{O}$ saturates unless, for every r, there is some polynomial which doesn't vanish on the whole of $\mathcal{O}$, but which vanishes on all the points of $\mathcal{O}$ with

$$\Omega(x_1 \cdots x_{s+1}) \leq r.$$

E.g. in one-dimensional space, $\mathcal{O} = \{2^k : k \in \mathbb{N}\}$ does not saturate, because $(X-1)\cdots(X-2^r)$ vanishes on the points with $\Omega(2^k) \leq r$, but not on $\mathcal{O}$. A version of that argument would work any time $\mathcal{O}$ is infinite and every $\Omega(x_1 \cdots x_{s+1}) \leq r$ defines a finite subset.

Previous work

When saturation was introduced by Sarnak (2008), $\mathcal{O}$ was an orbit of a finitely generated group. Later, the saturation of $V(\mathbb{Q})$ for irreducible varieties V was studied by various authors.

Previous work

When saturation was introduced by Sarnak (2008), $\mathcal{O}$ was an orbit of a finitely generated group. Later, the saturation of $V(\mathbb{Q})$ for irreducible varieties V was studied by various authors.

Some of the more general results:

- Sofos and Wang (2019) showed that if V is $\mathbb{Q}$-unirational then $V(\mathbb{Q})$ has finite saturation.
- Schindler and Sofos (2019) showed that $V(\mathbb{Q})$ has finite saturation if V is a smooth complete intersection of large codimension defined by a system of equations with the same degree.

Previous work

When saturation was introduced by Sarnak (2008), $\mathcal{O}$ was an orbit of a finitely generated group. Later, the saturation of $V(\mathbb{Q})$ for irreducible varieties V was studied by various authors.

Some of the more general results:

- Sofos and Wang (2019) showed that if V is $\mathbb{Q}$-unirational then $V(\mathbb{Q})$ has finite saturation.
- Schindler and Sofos (2019) showed that $V(\mathbb{Q})$ has finite saturation if V is a smooth complete intersection of large codimension defined by a system of equations with the same degree.

Bourgain, Gamburd and Sarnak (2010) mention an example for which, conjecturally, $V(\mathbb{Z})$ does not saturate.

Example: integral points on a conic

Bourgain, Gamburd and Sarnak (2010), see also Kontorovich (2014):

Example: integral points on a conic

Bourgain, Gamburd and Sarnak (2010), see also Kontorovich (2014):

Consider

$$X^2 + XY - Y^2 = 1$$

as an irreducible variety V in two-dimensional space. The integral points in $V(\mathbb{Z})$ are (F_{2n-1}, F_{2n}), where F_m is the mth Fibonacci number and we put

$$F_{-m} = (-1)^{m+1} F_m.$$

Example: integral points on a conic

Bourgain, Gamburd and Sarnak (2010), see also Kontorovich (2014):

Consider

$$X^2 + XY - Y^2 = 1$$

as an irreducible variety V in two-dimensional space. The integral points in $V(\mathbb{Z})$ are (F_{2n-1}, F_{2n}), where F_m is the mth Fibonacci number and we put

$$F_{-m} = (-1)^{m+1} F_m.$$

Conjecturally (see Bugeaud-Luca-Mignotte-Siksek 2010) we have

$$\omega(F_{2n}) \to \infty.$$

So $\Omega(x_2) \leq r$ defines a finite subset of $V(\mathbb{Z})$, and $V(\mathbb{Z})$ does not saturate.

Example: rational points on an elliptic curve

Consider the irreducible variety $E_{A,B}$ in two-dimensional space defined by

$$Y^2 = X^3 + Ax + B,$$

for some fixed $A, B \in \mathbb{Z}$. If $4A^3 + 27B^2 \neq 0$ this is an *elliptic curve*. Choose A, B so that $E_{A,B}(\mathbb{Q})$ is of rank 1 and is torsion-free.

Example: rational points on an elliptic curve

Consider the irreducible variety $E_{A,B}$ in two-dimensional space defined by

$$Y^2 = X^3 + Ax + B,$$

for some fixed $A, B \in \mathbb{Z}$. If $4A^3 + 27B^2 \neq 0$ this is an *elliptic curve*. Choose A, B so that $E_{A,B}(\mathbb{Q})$ is of rank 1 and is torsion-free.

According to a conjecture of Everest and King (2005), for any fixed r there should be only finitely many points

$$(X, Y) = (x_1/x_3, x_2, x_3)$$

of $E_{A,B}(\mathbb{Q})$ such that x_3 has r or fewer prime factors.

Example: rational points on an elliptic curve

Consider the irreducible variety $E_{A,B}$ in two-dimensional space defined by

$$Y^2 = X^3 + Ax + B,$$

for some fixed $A, B \in \mathbb{Z}$. If $4A^3 + 27B^2 \neq 0$ this is an *elliptic curve*. Choose A, B so that $E_{A,B}(\mathbb{Q})$ is of rank 1 and is torsion-free.

According to a conjecture of Everest and King (2005), for any fixed r there should be only finitely many points

$$(X, Y) = (x_1/x_3, x_2, x_3)$$

of $E_{A,B}(\mathbb{Q})$ such that x_3 has r or fewer prime factors.

But $E(\mathbb{Q})$ is infinite, so according to the conjecture it does not saturate.

Example: Hilbert 10

In the context of saturation, Sarnak (2008) also used the solution to Hilbert's Tenth Problem to show that the integer points of varieties may behave in strange ways. Developing this further we can get a counterexample to saturation.

Example: Hilbert 10

In the context of saturation, Sarnak (2008) also used the solution to Hilbert's Tenth Problem to show that the integer points of varieties may behave in strange ways. Developing this further we can get a counterexample to saturation.

The MDRP Theorem (Matiyasevich–Robinson–Davis–Putnam) shows that, for any recursively enumerable set $\mathcal{A} \subseteq \mathbb{Z}$, there is a variety V such that the set $\mathcal{A}$ is precisely the set of values taken by x_1 on the integral points $V(\mathbb{Z})$.

Example: Hilbert 10

In the context of saturation, Sarnak (2008) also used the solution to Hilbert's Tenth Problem to show that the integer points of varieties may behave in strange ways. Developing this further we can get a counterexample to saturation.

The MDRP Theorem (Matiyasevich–Robinson–Davis–Putnam) shows that, for any recursively enumerable set $\mathcal{A} \subseteq \mathbb{Z}$, there is a variety V such that the set $\mathcal{A}$ is precisely the set of values taken by x_1 on the integral points $V(\mathbb{Z})$.

Choosing $\mathcal{A}$ to be the powers of 2, we find that x_1 is unbounded on $V(\mathbb{Z})$ but takes only finitely many values with $\Omega(x_1) \leq r$. The same is true for some component W of V, giving an irreducible variety which does not saturate.

Example: Hilbert 10 (continued)

The MDRP Theorem (Matiyasevich–Robinson–Davis–Putnam) shows that, for any recursively enumerable set $\mathcal{A} \subseteq \mathbb{Z}$, there is a variety V such that the set $\mathcal{A}$ is precisely the set of values taken by x_1 on the integral points $V(\mathbb{Z})$.

Choosing $\mathcal{A}$ to be the powers of 2, we find that x_1 is unbounded on $V(\mathbb{Z})$ but takes only finitely many values with $\Omega(x_1) \leq r$. The same is true for some component W of V, giving an irreducible variety which does not saturate.

Example: Hilbert 10 (continued)

The MDRP Theorem (Matiyasevich–Robinson–Davis–Putnam) shows that, for any recursively enumerable set $\mathcal{A} \subseteq \mathbb{Z}$, there is a variety V such that the set $\mathcal{A}$ is precisely the set of values taken by x_1 on the integral points $V(\mathbb{Z})$.

Choosing $\mathcal{A}$ to be the powers of 2, we find that x_1 is unbounded on $V(\mathbb{Z})$ but takes only finitely many values with $\Omega(x_1) \leq r$. The same is true for some component W of V, giving an irreducible variety which does not saturate.

We can be more explicit using the explicit construction for the exponential function of Sun (1992), who greatly simplified and improved the explicit version of MDRP. This gives us an explicit (fairly long) equation in eight variables whose integer solutions satisfy

$$X_1 = 2^{2+X_2^2},$$

and hence a variety in eight-dimensional space whose integer points do not saturate.

Example: an extremely large subring of $\mathbb{Q}$

Theorem ((RM 2016))

Suppose that E is an elliptic curve and (technical condition) that $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \to \infty$ be a non-decreasing sequence. There is a set of primes S with natural density 1, and which in fact satisfies

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N},$$

such that $E(\mathbb{Z}[S^{-1}])$ does not saturate.

Concrete example 1

Theorem ((RM 2016))

Suppose (technical condition) $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \nearrow \infty$. There is a set of primes S such that $E(\mathbb{Z}[S^{-1}])$ does not saturate, with $\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}$.

Concrete example 1

Theorem ((RM 2016))

Suppose (technical condition) $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \nearrow \infty$. There is a set of primes S such that $E(\mathbb{Z}[S^{-1}])$ does not saturate, with $\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}$.

E.g. Let E be the elliptic curve $y^2 = x^3 - x^2$. Then $E(\mathbb{Q}) \cong \mathbb{Z}$ is generated by $P = (1, 1)$ where

$$2P = \left(\frac{13}{4}, -\frac{53}{8}\right)$$

and

$$3P = \left(\frac{5 \cdot 101}{3^4}, \frac{12203}{3^6}\right),$$

and so the hypothesis of the theorem is satisfied for both $\ell = 2$ and $\ell = 3$.

Concrete example 2

Theorem ((RM 2016))

Suppose (technical condition) $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \nearrow \infty$. There is a set of primes S such that $E(\mathbb{Z}[S^{-1}])$ does not saturate, with $\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}$.

Concrete example 2

Theorem ((RM 2016))

Suppose (technical condition) $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \nearrow \infty$. There is a set of primes S such that $E(\mathbb{Z}[S^{-1}])$ does not saturate, with $\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}$.

E.g. Let E be the elliptic curve

$$y^2 + xy + y = x^3 - x^2.$$

Then $E(\mathbb{Q}) \cong \mathbb{Z}$ is generated by $P = (0,0)$, and we have

$$4P = \left(-\frac{1}{4}, -\frac{5}{8}\right),$$

so the theorem applies with $\ell = 2$ and $\mathcal{O} = E(O_{\mathbb{Q},S})$.

Example: an extremely large subring of $\mathbb{Q}$

Theorem ((RM 2016))

Suppose that E is an elliptic curve and (technical condition) that $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \to \infty$ be a non-decreasing sequence. There is a set of primes S with natural density 1, and which in fact satisfies

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N},$$

such that $E(\mathbb{Z}[S^{-1}])$ does not saturate.

Example: an extremely large subring of $\mathbb{Q}$

Theorem ((RM 2016))

Suppose that E is an elliptic curve and (technical condition) that $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \to \infty$ be a non-decreasing sequence. There is a set of primes S with natural density 1, and which in fact satisfies

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N},$$

such that $E(\mathbb{Z}[S^{-1}])$ does not saturate.

Proof idea: the "indicator primes" construction of Cornelissen and Shlapentokh (2009).

Example: an extremely large subring of $\mathbb{Q}$

Theorem ((RM 2016))

Suppose that E is an elliptic curve and (technical condition) that $E(\mathbb{Q})$ is torsion-free of rank 1, generated by P such that $\ell \mid x_3(\ell^M P)$ for some $M \in \mathbb{N}$ and some prime ℓ. Let $A_N \to \infty$ be a non-decreasing sequence. There is a set of primes S with natural density 1, and which in fact satisfies

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N},$$

such that $E(\mathbb{Z}[S^{-1}])$ does not saturate.

Proof idea: the "indicator primes" construction of Cornelissen and Shlapentokh (2009).

We will choose the set S to contain all primes except one indicator prime for each point nP, where $n \notin \{\ell^k\}$ and $x_3(nP)$ is not smooth.

Question: an even larger set of semi-integral points

In the theorem, $E(\mathbb{Z}[S^{-1}])$ does not saturate, where

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}.$$

Question: an even larger set of semi-integral points

In the theorem, $E(\mathbb{Z}[S^{-1}])$ does not saturate, where

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}.$$

If S contained all but finitely many primes, an argument of Robinson (1949) would yield a variety V such that $V(\mathbb{Q})$ does not saturate. But this seems beyond reach for now.

Question: an even larger set of semi-integral points

In the theorem, $E(\mathbb{Z}[S^{-1}])$ does not saturate, where

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}.$$

If S contained all but finitely many primes, an argument of Robinson (1949) would yield a variety V such that $V(\mathbb{Q})$ does not saturate. But this seems beyond reach for now.

A set of *semi-integral points* on V is any subset of $V(\mathbb{Q})$ defined by conditions of the form

$$v_p(x_{s+1}) \in E_p,$$

where $p^{v_p(x)}$ is the exact power of p dividing x, and E_p is some sequence of sets.

Question: an even larger set of semi-integral points

In the theorem, $E(\mathbb{Z}[S^{-1}])$ does not saturate, where

$$\#\{p \leq N : p \notin S\} \leq \sqrt{A_N \log N}.$$

If S contained all but finitely many primes, an argument of Robinson (1949) would yield a variety V such that $V(\mathbb{Q})$ does not saturate. But this seems beyond reach for now.

A set of *semi-integral points* on V is any subset of $V(\mathbb{Q})$ defined by conditions of the form

$$v_p(x_{s+1}) \in E_p,$$

where $p^{v_p(x)}$ is the exact power of p dividing x, and E_p is some sequence of sets.

Question: Find a variety V and a sequence e_p such that the semi-integral points of V do not saturate, and such that the set

$$\{x : v_p(x) \in E_p, \ \forall p\}$$

is bigger than $\mathbb{Z}[S^{-1}]$ (in some reasonable sense).

Question: thin sets

A subset of $V(\mathbb{Q})$ is a *type II thin set* if it is contained in $\pi(W(\mathbb{Q}))$, where W is an irreducible variety and $\pi : W \to V$ is a generically finite, dominant morphism with degree ≥ 2.

To be very sloppy indeed, a type II thin set is one that is parametrised by nonlinear polynomials. E.g. the set of rational squares is a thin subset of $\mathbb{Q}$.

Question: thin sets

A subset of $V(\mathbb{Q})$ is a *type II thin set* if it is contained in $\pi(W(\mathbb{Q}))$, where W is an irreducible variety and $\pi : W \to V$ is a generically finite, dominant morphism with degree ≥ 2.

To be very sloppy indeed, a type II thin set is one that is parametrised by nonlinear polynomials. E.g. the set of rational squares is a thin subset of $\mathbb{Q}$.

A subset of $V(\mathbb{Q})$ is *thin* in the sense of Serre if, after removing finitely many type II thin sets, what is left is not Zariski dense.

Question: thin sets

A subset of $V(\mathbb{Q})$ is a *type II thin set* if it is contained in $\pi(W(\mathbb{Q}))$, where W is an irreducible variety and $\pi : W \to V$ is a generically finite, dominant morphism with degree ≥ 2.

To be very sloppy indeed, a type II thin set is one that is parametrised by nonlinear polynomials. E.g. the set of rational squares is a thin subset of $\mathbb{Q}$.

A subset of $V(\mathbb{Q})$ is *thin* in the sense of Serre if, after removing finitely many type II thin sets, what is left is not Zariski dense.

Question: find an irreducible variety V such that $V(\mathbb{Q})$ is not thin, but for every r the subset defined by $\Omega(x_1 \cdots x_{s+1}) \leq r$ is thin.

Question: thin sets

A subset of $V(\mathbb{Q})$ is a *type II thin set* if it is contained in $\pi(W(\mathbb{Q}))$, where W is an irreducible variety and $\pi : W \to V$ is a generically finite, dominant morphism with degree ≥ 2.

To be very sloppy indeed, a type II thin set is one that is parametrised by nonlinear polynomials. E.g. the set of rational squares is a thin subset of $\mathbb{Q}$.

A subset of $V(\mathbb{Q})$ is *thin* in the sense of Serre if, after removing finitely many type II thin sets, what is left is not Zariski dense.

Question: find an irreducible variety V such that $V(\mathbb{Q})$ is not thin, but for every r the subset defined by $\Omega(x_1 \cdots x_{s+1}) \leq r$ is thin.

Motivation: Using Koenigsmann (2016), for any V one can construct a variety W and a morphism π such that

$$V(\mathbb{Q}) \setminus \pi(W(\mathbb{Q})) = V(\mathbb{Z}).$$

But: π is not generically finite (W has bigger dimension than V).